SAMPLING THEOREMS FOR NON BANDLIMITED SIGNALS: THEORETICAL IMPACT AND PRACTICAL APPLICATIONS

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Abstract.[1] In recent years many of the results for band limited sampling have been extended to the case of non bandlimited signals. These recent extensions have been found to be useful in digital signal processing applications such as image interpolation, equalization of communication channels, and in multiresolution computation. In this lecture we give a brief overview of some of these ideas.

1. INTRODUCTION

The uniform sampling theorem for bandlimited signals was introduced by brilliant minds more than 6 decades ago, and is the backbone of all digital signal processing and digital communication systems today. The theorem has been extended in many directions. This includes nonuniform sampling and derivative sampling of bandlimited signals [6], and more general multichannel sampling [11]. In recent years many of these results have been extended to the case of non bandlimited signals. These recent extensions have been found to be useful in applications such as image interpolation, equalization of communication channels, and in multiresolution computation. In this lecture we give a brief overview of some of these ideas. Detailed overviews can be found in [25], [26], [20], [21], and [30].[2]

Notations. Unless mentioned otherwise, all notations are as in [22]. The subscript d as in $x_d(n)$ is used for discrete time quantities if there is potential confusion. Bold faced letters denote matrices and vectors. The term σ-BL refers to signals that are bandlimited to $|\omega| < \sigma$ (i.e., Fourier transform is zero outside). We use the notations $[x(n)]_{\downarrow M}$ and $[X(z)]_{\downarrow M}$ to denote the decimated version $x(Mn)$ and its z-transform. The expanded version

$$\begin{cases} x(n/M) & n = \text{mul of } M, \\ 0 & \text{otherwise} \end{cases}$$

is similary denoted by $[x(n)]_{\uparrow M}$, and its z-transform $X(z^M)$ denoted by $[X(z)]_{\uparrow M}$. In situations where the z-transform does not exist in the conventional sense (e.g., ideal filters), the notation z stands for $e^{j\omega}$ so that $F(z)$ is the frequency response $F(e^{j\omega})$. Any transfer function $F(z)$ can be written in the form

$$F(z) = \sum_{k=0}^{M-1} z^k R_k(z^M)$$

called the (type 2) polyphase representation [22].

2. SIGNAL MODELS

If $x(t)$ is π-BL we can recover it from integer spaced samples $x(n)$ (Shannon or Nyquist sampling). This corresponds to a sampling rate $\Omega_s = 2\pi$ radians/sec (or 1 Hz). If a signal is not bandlimited, can we still recover it from samples? The answer depends on what other apriori information we have. For example, suppose we have the knowledge that $x(t)$ has the form

$$x(t) = \sum_{k=-\infty}^{\infty} c(k)\phi(t-k) \quad \text{(signal model)} \qquad (1)$$

where $\phi(t)$ is a **known** function (not necessarily bandlimited). Suppose $\phi(t)$ has the zero-crossing property $\phi(n) = \delta(n)$, i.e.,

$$\phi(0) = 1, \quad \phi(n) = 0 \text{ for other integer } n \qquad (2)$$

as demonstrated in Fig. 1(a). This implies in particular that

$$x(t) = \sum_k x(k)s(t-k) \qquad (3)$$

where $s(t) = \phi(t)$. Thus reconstruction from samples $x(n)$ has been possible inspite of the aliasing effects due to non bandlimitedness. A function $\phi(t)$ satisfying the zero-crossing property (2) is also referred to as a **Nyquist(1)** function in the literature. The argument "(1)" signifies that the zero crossings are separated by one unit of time. A special case is the example where $\phi(t)$ is the sinc funtion (Fig. 1(b))

$$\phi(t) = \frac{\sin \pi t}{\pi t} \qquad (4)$$

Since this is π-BL, the sum (1) is also π-BL. The sinc function is Nyquist(1), so the reconstruction formula (3) holds. This is the familiar Shannon reconstruction.

[1] Work supported in parts by the NSF grant MIP 0703755 and ONR grant N00014-99-1-1002.

[2] References [25, 26, 28, 30] can be found under "publications" at http://www.systems.caltech.edu/EE/Groups/dsp/.

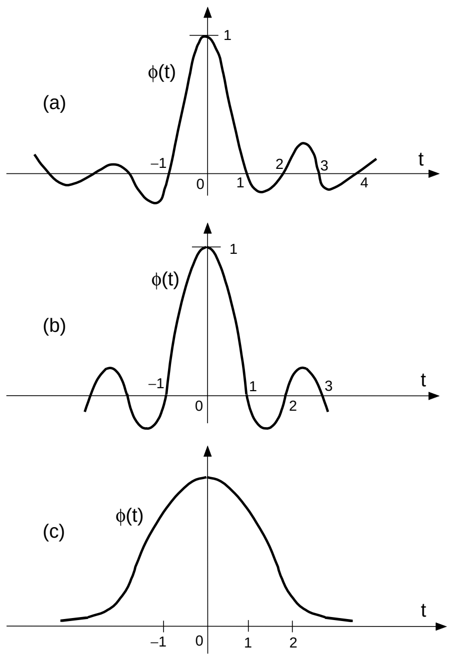

Fig. 1. Examples of the function $\phi(t)$ in the signal model. (a) Nyquist(1) function, (b) $\sin(\pi t)/\pi t$, and (c) non Nyquist(1) function.

If a function can be represented as in Eq. (1) where $\phi(t)$ is *not Nyquist* (Fig. 1(c)), can we still reconstruct $x(t)$ from samples $x(n)$? The answer is in the affirmative for a large class of $\phi(t)$. From Eq. (1) we see that the samples of $x(t)$ are given by

$$x_d(n) \stackrel{\Delta}{=} x(n) = \sum_{k=-\infty}^{\infty} c(k)\phi(n-k) \qquad (5)$$

which is nothing but a discrete-time convolution equation. Denoting the discrete time Fourier transforms of the sequences $x(n)$, $c(n)$, and $\phi(n)$ by $X_d(e^{j\omega}), C(e^{j\omega})$ and $\Phi_d(e^{j\omega})$ we get $X_d(e^{j\omega}) = C(e^{j\omega})\Phi_d(e^{j\omega})$. Note that

$$\Phi_d(e^{j\omega}) = \sum_n \Phi(\omega + 2\pi n)$$

If $\Phi_d(e^{j\omega}) \neq 0$ for all ω we can write

$$C(e^{j\omega}) = \frac{X_d(e^{j\omega})}{\Phi_d(e^{j\omega})} \qquad (6)$$

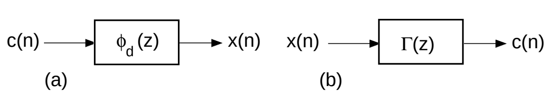

Fig. 2. (a) Samples of $x(t)$ modelled as the output of a digital filter, and (b) reconstruction of $c(n)$ from the samples using the inverse digital filter.

That is, we can identify $c(n)$ from $x(n)$ using

$$c(n) = \sum_k x(k)\gamma(n-k) \qquad (7)$$

where $\gamma(n)$ is the **convolutional inverse** of $\phi(n)$ i.e., its Fourier transform $\Gamma(e^{j\omega}) = 1/\Phi_d(e^{j\omega})$. Recovery of $x(t)$ for all t can then be done using (1). Notice that the sequences $x(n)$ and $c(n)$ are related by digital filters as shown in Fig. 2.

2.1. Remarks

1. ***Undersampling a wideband signal.*** As a special case of the signal model, suppose $\phi(t)$ is 2π-BL, that is, bandlimited to $-2\pi < \omega < 2\pi$. Then $x(t)$ is also 2π-BL. In this case, the Nyquist rate would be 4π, implying the sample spacing $T = 1/2$. The preceding theory shows that the wider sample spacing of $T = 1$ is actually sufficient! As a specific example, if $\Phi(j\omega)$ is real and positive in $-2\pi < \omega < 2\pi$ then $\Phi_d(e^{j\omega})$ is surely nonzero and we can reconstruct $x(t)$ from these lower rate samples.

2. ***Lack of shift invariance.*** For fixed $\phi(t)$, let V_0 denote the space of all signals which can be represented as in (1) for appropriate finite-energy $c(k)$. When $\phi(t)$ is the sinc function we know that any shifted version of $x(t)$ (e.g., $x(t-0.1)$) also belongs to the space V_0 because time-shift does not affect bandlimitedness. For arbitrary $\phi(t)$ however, even though reconstruction from samples is often possible, the shifted versions of $x(t)$ do not in general belong to the same space V_0.

3. ***Reconstruction filter***. Substituting for $c(k)$ from (7) into (1) and simplifying we get

$$x(t) = \sum_{i=-\infty}^{\infty} x(i)s(t-i)$$

where $s(t) = \sum_{m=-\infty}^{\infty} \gamma(m)\phi(t-m)$ is the reconstruction filter. Thus, we simply pass the samples $x(n)$ through the continuous-time filter with impulse respose $s(t)$ as in standard D/A conversion (Fig. 3).

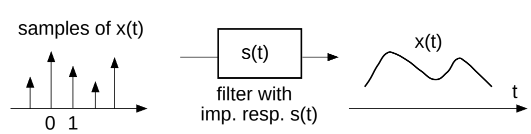

Fig. 3. Reconstruction of $x(t)$ from samples.

From the preceding definition we can show that the reconstruction filter $s(t)$ has the Nyquist(1) property $s(n) = \delta(n)$ similar to the sinc function $\sin \pi t/\pi t$.

2.2. Application in interpolation

The preceding discussion also shows this: given a discrete time signal $x_d(n)$ and an arbitrary function $\phi(t)$, we can **almost always** assume that $x_d(n)$ can be written in the form (5) for appropriate choice of $c(k)$, the only theoretical condition being that $\Phi_d(e^{j\omega}) \neq 0$ for all ω. Thus we can regard $x_d(n)$ as samples of the continuous time signal

$$x(t) \stackrel{\Delta}{=} \sum_k c(k)\phi(t-k).$$

While true for almost any $\phi(t)$, this viewpoint is especially useful for certain choices of $\phi(t)$. If $\phi(t)$ has smoothness properties such as a certain degree of differentiability everywhere, then we can use this to generate an interpolated version of $x_d(n)$. For example, a 256×256 image can be displayed as a 512×512 image in this way (interpolation by two). Smoothness of $\phi(t)$ usually ensures that the interpolated result is visually pleasing.

To see how the model can be used for interpolation notice that the samples of $x(t)$ at a finer spacing $1/L$ are given by

$$x(n/L) = \sum_{k=-\infty}^{\infty} c(k)\phi(\frac{n-kL}{L}) = \sum_{k=-\infty}^{\infty} c(k) f_L(n-kL) \quad (8)$$

where $f_L(i) = \phi(i/L)$. The quantity on the right hand side of (8) can be interpreted as the output of a digital interpolation filter $F_L(z) = \sum_n f_L(n)z^{-n}$ as shown in Fig. 4(a). Thus we can reconstruct the finer samples $x(n/L)$ from $x(n)$ as shown in Fig. 4(b): first pass $x(n)$ through the digital prefilter $\Gamma(z) = 1/\Phi_d(z)$ to get $c(n)$. Then use the L-fold expander and interpolation filter $F_L(z)$ to get $x(n/L)$. We see that the interpolation from $x(n)$ to $x(n/L)$ can be done entirely digitially.

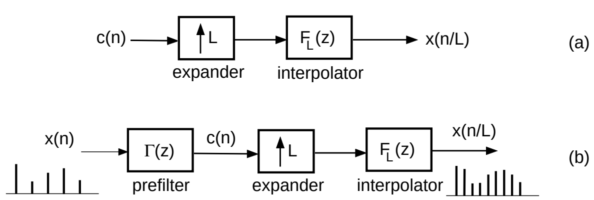

Fig. 4. Interpolation of a signal $x(n)$ with digital filters. The signal is assumed to have a continuous time model $x(t) = \sum_k c(k)\phi(t-k)$.

Spline interpolation. In image processing, $\phi(t)$ is typically chosen as a B-spline (especially cubic spline) rather than a bandlimited sinc function. If $\phi(t)$ is an Nth order B-spline then $x(t)$ is also an Nth order spline, and both $\phi(t)$ and $x(t)$ are continuously differentiable $N-1$ times [12]. The spline $\phi(t)$ has finite duration so that $\Phi_d(z)$ is FIR and the inverse filter $\Gamma(z)$ IIR. Since $\phi(t)$ is symmetric about its midpoint, $\Phi_d(z)$ has coefficient-symmetry. The filter $\Gamma(z)$ therefore has poles both inside and outside the unit circle. So a **noncausal** implementation has to be used to ensure filter **stability** [10].

Sampling theorems based on the signal model (1) were observed independently by many authors, e.g., Walter [32] and Daubechies [4]. While the history of interpolation theory is rather long, some of the fairly recent advances made by Unser, Aldroubi, and their colleagues have been most influential in this area [17, 19,1]. For example, spline interpolation of images has been made very simple and attractive. Such interpolation can now be done with a simple combination of FIR and IIR filters [17], and sometimes with very inexpensive FIR filters alone [27]. Many image processing operations such as rotation, least squares smoothing, and edge detection have been simplified by this idea [19].

3. BIORTHOGONAL PARTNERS

We now review the notion of biorthogonal partners introduced in [25]. The usefulness of this idea lies in the fact that it naturally makes a connection between interpolation theory and sampling theorems. This makes it useful for application in least squares signal modelling. In fact biorthogonal partners arise in digital communications in a rather hidden form. This includes fractionally spaced equalizers (FSE) [15] and filter bank precoders [14,31]. The multi channel version of the biorthogonal partner idea has further applications, for example, in the equalization of multiuser channels [28], [29]. The complete theory of biorthogonal partners can be found in the references given above, so we shall be brief here. At this point please review notations in Sec. 1.

Definition 1. *Biorthogonal partners.* Two transfer functions $F(z)$ and $H(z)$ are said to be biorthogonal partners of each other with respect to an integer M if

$$\Big[H(z)F(z)\Big]_{\downarrow M} = 1$$

This is equivalent to the statement that the product filter $P(z) \stackrel{\Delta}{=} H(z)F(z)$ is Nyquist(M), that is, its impulse response $p(n)$ has the zero-crossing property $p(Mn) = \delta(n)$. as shown in Fig. 5. $\diamondsuit$

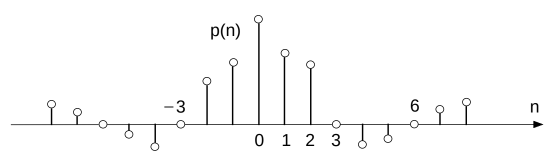

Fig. 5. Nyquist(M) property of the product fillter $P(z)$ demonstrated for $M = 3$.

Referring to Fig. 6, it can be shown that the transfer function from $c(n)$ to $d(n)$ is given precisely by the quantity $[H(z)F(z)]_{\downarrow M}$. Thus, biorthogonal partnership means that this system is an identity system $[d(n) = c(n)]$. In other words, the decimation operation shown in the figure is the inverse of the interpolation operation.

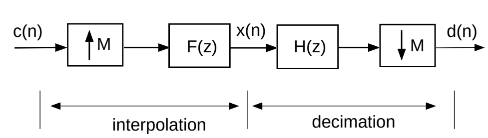

Fig. 6. Interpreting biorthogonal partnership using block diagrams.

3.1. Existence and FIR Solution

Given a transfer function $F(z)$, does a biorthogonal partner exist? Is it unique? Well, the inverse $1/F(z)$ always serves as a biorthogonal partner (with respect to any M) as long as it exits (i.e., as long as $F(e^{j\omega}) \neq 0$ for any ω). But the partner is not unique. For example,

$$H(z) = \frac{1}{[F(z)]_{\downarrow M \uparrow M}}$$

is a valid partner assuming the denominator does not vanish on the unit circle. The next result, proved in [25], addresses the existence of FIR partners.

Theorem 1. *FIR biorthogonal partner.* Let $F(z)$ be FIR, that is, $F(z) = \sum_{n=n_0}^{n_1} f(n)z^{-n}$ for finite n_0, n_1. Express it in the polyphase form $F(z) = \sum_{k=0}^{M-1} z^k R_k(z^M)$. Then there exists an FIR filter $H(z)$ such that

$$[H(z)F(z)]_{\downarrow M} = 1$$

if and only if the greatest common divisor of the M polyphase components $\{R_k(z)\}$ is trivial (i.e., has the form cz^{-N}). $\diamondsuit$

The condition in the theorem is equivalent to the statement that the M polyphase components do not share a common zero. For example suppose $F(z) = 1 + 4z^{-1} + z^{-2}$. Then the obvious biorthogonal partner $H(z) = 1/F(z)$ is IIR with poles both inside and outside the unit circle. So we cannot have a causal and stable implementation. In this example we can actually find an FIR partner for $M = 2$. For this note that the two polyphase components of $F(z)$ are $R_0(z) = 1 + z^{-1}$ and $R_1(z) = 4z^{-1}$ so that the condition of the theorem is satisfied. Indeed the FIR filter $H(z) = z^2 - 0.25z^3$ can be verified to be a partner for $M = 2$.

In what context do biorthogonal partners arise? Readers familiar with maximally decimated filter banks with analysis filters $\{H_k(z)\}$ and synthesis filters $\{F_k(z)\}$ will recall that the system has the perfect reconstruction property if and only if $[H_k(z)F_m(z)]_{\downarrow M} = \delta(k - m)$. This means in particular that the filters $H_k(z)$ and $F_k(z)$ are biorthogonal partners for each k. So the property arises very naturally in filter bank theory. A more appropriate situation for this lecture is in signal interpolation.

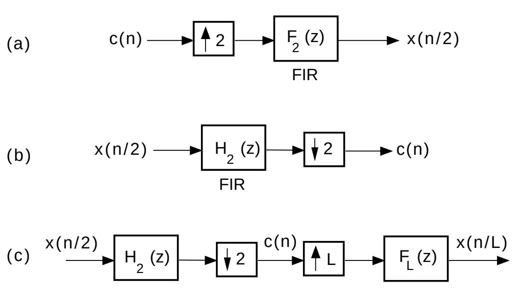

Fig. 7. (a) Model for the oversampled signal $x(n/2)$, (b) the inverse which generates $c(n)$ from the oversampled version $x(n/2)$, and (c) construction of interpolated version $x(n/L)$ from the oversampled version $x(n/2)$ using only FIR filters.

3.2. Oversampling And Interpolation

Let us return to the interpolation formula given in Eq. (8). Figure 4 shows an implementation of this formula. As explained in Sec. 2.2, if $\phi(t)$ is a spline then the prefilter $\Gamma(z)$ is an unstable IIR filter unless we use noncausal implementation methods.

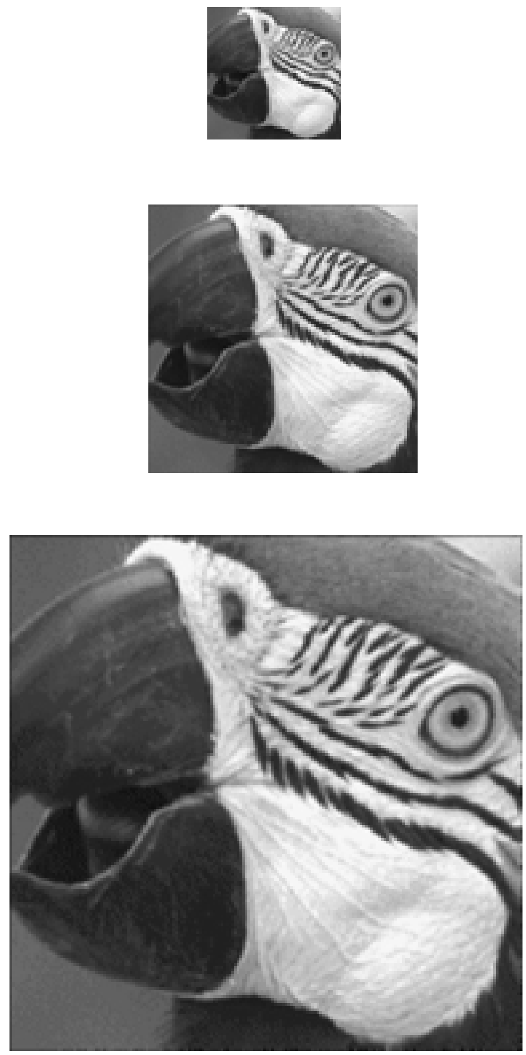

Fig. 8. The top image $x(n)$ is a 128×128 portion of the parrot image. Middle image $x(n/2)$ is obtained using a cubic spline model. This computation requires an IIR filter $\Gamma(z)$ (Fig. 4). Bottom image $x(n/4)$ is obtained from $x(n/2)$ using FIR filtering (Fig. 7(c) with $L = 4$).

Imagine now that $x(t)$ has been oversampled to obtain $x(n/2)$. From this signal we will show how $x(n/L)$ can be obtained by FIR filtering alone. From Fig. 4(a) we see that $x(n/2)$ can be represented as in Fig. 7(a) where $F_2(z)$ has impulse response $f_2(n) = \phi(n/2)$. Since $\phi(t)$ has finite duration, $F_2(z)$ is FIR. Let $H_2(z)$ be a biorthogonal partner of $F_2(z)$ with respect to the integer $M = 2$. Then we can construct $c(n)$ from $x(n/2)$ as shown in Fig.

7(b). For the example where $\phi(t)$ is the cubic spline, it is demonstrated in [25] that there exists an FIR partner $H_2(z)$. Indeed, in this case it can be verified that $F_2(z)$ is

$$\frac{z^{-1}+8z^{-2}+23z^{-3}+32z^{-4}+23z^{-5}+8z^{-6}+z^{-7}}{48}$$

and its FIR biorthogonal partner is

$$H_2(z)=\frac{3z^{10}-24z^{9}+68z^{8}-88z^{7}+47z^{6}}{6}$$

Thus $c(n)$ can be computed from $x(n/2)$ using FIR filtering. From $c(n)$ we can then compute $x(n/L)$ for any L using the FIR filter $F_L(z)$. The complete FIR system is shown in Fig. 7(c). An image interpolation example is shown in Fig. 8. Please see [25] for all further details.

3.3. Multiresolution Theory And Sampling

Consider again signals of the form $x(t)=\sum_k c(k)\phi(t-k)$ where $\phi(t)$ has finite duration. If $c(k)$ has to be computed from the samples $x(n)$, this requires the IIR filter $1/\Phi_d(e^{j\omega})$. We just showed that if the oversampled version $x(n/2)$ is available then we can compute $c(k)$ using only FIR filters. This is an attractive alternative to what is conventionally done in multiresolution analysis [3] to compute $c(k)$ from a highly oversampled version. To appreciate the difference between the above FIR construction and the conventional "highly oversampled construction", we now give a brief review of the latter.

Assuming $c(k)$ is in ℓ_2 and $\phi(t)\in L^2$, the set of functions $x(t)=\sum_k c(k)\phi(t-k)$ forms a subspace $V_0\subset L^2$. This subspace is spanned by the integer-shifted versions $\{\phi(t-k)\}$ (Fig. 9(a)). Now consider the squeezed version $\phi(2t)$ and its shifted versions $\phi(2t-k)$ sketched in Fig. 9(b). This set also spans a subspace $V_1\subset L_2$. In multiresolution theory $\phi(t)$ is chosen such that $V_0\subset V_1$. In particular $\phi(t)$ is a linear combination of $\phi(2t-k)$, that is,

$$\phi(t)=2\sum_m f(m)\phi(2t-m) \qquad (9)$$

This is the familiar **dilation equation** [4], [8] and translates in the Fourier domain to $\Phi(j\omega)=F(e^{j\omega/2})\Phi(j\omega/2)$. By repeating this idea we see that $x(t)$ belongs to the space V_J spanned by $\{\phi(2^Jt-k)\}$ for any integer $J\geq 0$, that is,

$$x(t)=\sum_k c_J(k)2^{J/2}\phi(2^Jt-k) \qquad (10)$$

The multiresolution coefficients $c_J(k)$ at scale J reduce to the usual $c(k)$ for $J=0$. The constant $2^{J/2}$ merely ensures that the scaled basis functions $\{2^{J/2}\phi(2^Jt-k)\}$ have the same energy for all J. Fig. 9(c) shows $x(t)$ and several shifted versions $\phi(2^Jt-k)$ for large J. We shall now argue that the samples $x(2^{-J}k)$ are approximately proportional to $c_J(k)$. Since $x(2^{-J}n)=\sum_k c_J(k)2^{J/2}\phi(n-k)$, the sequence $x(2^{-J}n)$ is the output of the digital filter $\phi(n)$ in response to the input $2^{J/2}c_J(n)$. Thus, except for a constant multiplier, $c_J(n)$ is the output of the inverse filter $\gamma(n)$ in response to the input $x(2^{-J}n)$. If J is large enough, then $x(2^{-J}n)$ is nearly constant in the region where $\gamma(n)$ is significant. Thus the output $c_J(n)$ is also slowly varying, and is nearly proportional to the input, that is, $c_J(k)\approx\alpha_J x(2^{-J}k)$. If the oversampling factor 2^J is large enough, this estimate of $c_J(k)$ is very good. The beauty of the dilation equation is that it allows us to compute the multiresolution coefficients at lower scales

$$c_{J-1}(k),\ c_{J-2}(k),\ \ldots,\ c_0(k)$$

successively from $c_J(k)$, and thereby identify $c(k)=c_0(k)$. To see this, substitute (9) into (10). Then

$$x(t)=\sum_\ell \underbrace{\sum_k \sqrt{2}c_J(k)f(\ell-2k)}_{c_{J+1}(\ell)}\ 2^{(J+1)/2}\phi(2^{J+1}t-\ell)$$

That is, $c_{J+1}(n)=\sqrt{2}\sum_k c_J(k)f(n-2k)$, which shows that we can go from scale J multiresolution coefficients $c_J(n)$ to the scale $J+1$ coefficients $c_{J+1}(n)$ by using an interpolation filter as shown in Fig. 10(a), where $F(z)=\sum_n f(n)z^{-n}$. If $\sqrt{2}F(z)$ has a **biorthogonal partner** $\sqrt{2}H(z)$ we can also go from scale $J+1$ to J by using the decimation filter of Fig. 10(b). This shows that we can compute the coefficients $c_i(n)$ for all lower scales using the **multistage decimation** system shown in Fig. 10(c).

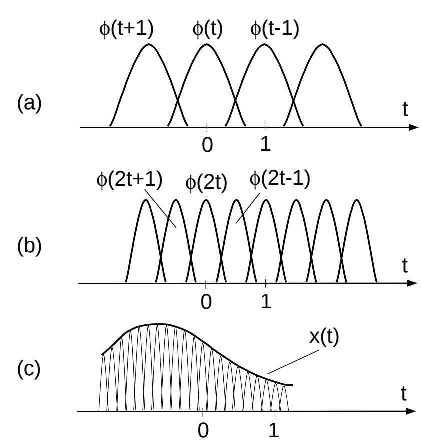

Fig. 9. (a) The function $\phi(t)$ and its integer shifted versions, (b) the squeezed function $\phi(2t)$ and its shifted versions $\phi(2t-k)$, and (c) several shifted and weighted versions $c_J(k)2^{J/2}\phi(2^Jt-k)$ for $2^J=8$, shown along with $x(t)=\sum_k c_J(k)2^{J/2}\phi(2^Jt-k)$.

If $F(z)$ is FIR with coprime polyphase components $R_0(z)$ and $R_1(z)$ (where $F(z)=R_0(z^2)+zR_1(z^2)$) then we can find an FIR filter $H(z)$ to implement Fig. 10(c). Notice finally that if $\{\eta_k(n)\}=\{\sqrt{2}f(n-2k)\}$ is an orthonormal

[3] We refer the reader to Mallat's book [8] for an excellent treatment of multiresolution theory.

set then $h(n) = f^*(-n)$ works in the preceding scheme. In order for the above oversampling strategy to yield good results, we have to make the oversampling factor large so that the approximation of $c_J(n)$ is good. Compare this with the method of Sec. 3.2 which yields exact results and requires oversampling only by a factor of two, and the nonuniform method to be presented in Sec. 4, which yields exact results with no oversampling at all.

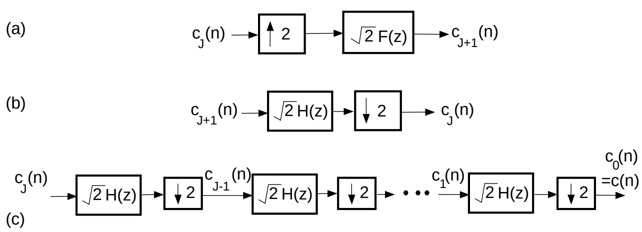

Fig. 10. Details of conventional multiresolution computation. (a) Representation of $c_{J+1}(n)$, (b) computation of $c_J(n)$ from $c_{J+1}(n)$, where $\sqrt{2}H(z)$ is a biorthogonal partner of $\sqrt{2}F(z)$, and (c) multistage decimation circuit for computation of the coefficients $c_i(n)$ for all lower-level scales, upto $c_0(n) = c(n)$.

3.4. Oversampling In Digital Communications

Another scenario where biorthogonal partners arise is in channel equalization. Since the idea is applicable to vector channels as well, we describe it using matrix-vector notations. Consider a transfer matrix $\mathbf{F}(z)$ representing, for example, a multi input multi output channel.

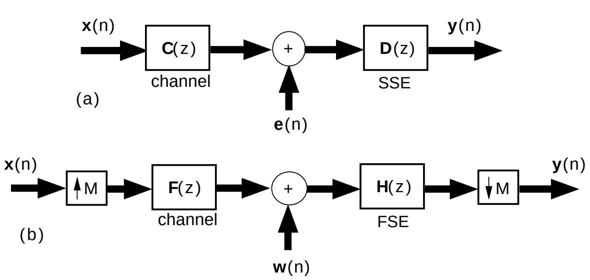

Fig. 11. Application of biorthogonal partners in channel equalization. (a) Symbol spaced equalizer (SSE), and (b) fractionally spaced equalizer (FSE). Please see text.

We say that a transfer matrix $\mathbf{H}(z)$ is a (left) biorthogonal partner[4] of $\mathbf{F}(z)$ with respect to an integer M if [28, 29]

$$[\mathbf{H}(z)\mathbf{F}(z)]_{\downarrow M} = \mathbf{I}, \tag{11}$$

where the notation $[\mathbf{C}(z)]_{\downarrow M}$ means that each matrix element $C_{ij}(z)$ is decimated (in time) by M. Imagine that a vector sequence $\mathbf{x}(n)$ of symbols is input to a channel which can be approximated with an FIR matrix $\mathbf{C}(z)$, and additive noise $\mathbf{e}(n)$ (Fig. 11(a)). (Here $\mathbf{C}(z)$ and $\mathbf{e}(n)$ are sampled versions from the continuous-time channel). If we wish to design an ideal zero-forcing equalizer $\mathbf{D}(z)$, it would have to be a left inverse (i.e., $\mathbf{D}(z)\mathbf{C}(z) = \mathbf{I}$). Such an inverse may not be FIR, and may have poles close to unit circle (it may not even be stable). This could lead to severe noise amplification. Notice that $\mathbf{D}(z)$ is a symbol-spaced equalizer or **SSE**.

[4]Right biorthogonal partners can be defined in a similar way but we shall not use them here.

Consider next the scheme of Fig. 11(b). Here $\mathbf{F}(z)$ and $\mathbf{w}(n)$ represent M-fold oversampled versions of the channel and its noise. The system $\mathbf{H}(z)$ is a fractionally spaced equalizer or **FSE** for the channel. If this is chosen as a left biorthogonal partner of $\mathbf{F}(z)$ (with respect to M) then we have a zero-forcing equalizer because the system function from $\mathbf{x}(n)$ to $\mathbf{y}(n)$ is $[\mathbf{H}(z)\mathbf{F}(z)]_{\downarrow M} = \mathbf{I}$. FIR solutions for $\mathbf{H}(z)$ are more likely to exist in this case than in the SSE scheme of Fig. 11(a) as we can readily demonstrate [28]. The FIR FSE is not only convenient, it also results in less noise amplification.

We can exploit the nonuniqueness of FIR biorthogonal partners to further improve the performance. Assume $\mathbf{F}(z)$ is causal and FIR. In [28] we have established the conditions for the existence of FIR biorthogonal partners. For example consider $M = 2$ and write $\mathbf{F}(z)$ in polyphase form [22] $\mathbf{F}(z) = \mathbf{F}_0(z^2) + z\mathbf{F}_1(z^2)$. Then we can show the following [28, 29]:

Theorem 2. *MIMO FIR partner.* An FIR partner for the FIR filter $\mathbf{F}(z)$ exists if and only if the polyphase components $\mathbf{F}_0(z)$ and $\mathbf{F}_1(z)$ are right-coprime, that is, if and only if the greatest common right divisor (grcd), denoted $\mathbf{R}(z)$, is a unimodular matrix [22]. $\diamondsuit$

Assume this condition is satisfied, and express the partner in polyphase form $\mathbf{H}(z) = \mathbf{H}_0(z^2) + z^{-1}\mathbf{H}_1(z^2)$. Then we can construct infinitely many FIR solutions from this! The crucial fact that leads to this observation is that when $\mathbf{F}_0(z)$ and $\mathbf{F}_1(z)$ are right-coprime, there exists [22] a unimodular matrix $\mathbf{U}(z)$ such that

$$\underbrace{\begin{bmatrix} \mathbf{U}_{11}(z) & \mathbf{U}_{12}(z) \\ \mathbf{U}_{21}(z) & \mathbf{U}_{22}(z) \end{bmatrix}}_{\mathbf{U}(z)} \begin{bmatrix} \mathbf{F}_0(z) \\ \mathbf{F}_1(z) \end{bmatrix} = \begin{bmatrix} \mathbf{R}(z) \\ \mathbf{0} \end{bmatrix}$$

It can then be shown that if the polyphase components of $\mathbf{H}(z)$ are modified to

$$\widehat{\mathbf{H}}_0(z) = \mathbf{H}_0(z) + \mathbf{A}(z)\mathbf{U}_{21}(z)$$
$$\widehat{\mathbf{H}}_1(z) = \mathbf{H}_1(z) + \mathbf{A}(z)\mathbf{U}_{22}(z) \tag{12}$$

the result $\widehat{\mathbf{H}}(z) = \widehat{\mathbf{H}}_0(z^2) + z^{-1}\widehat{\mathbf{H}}_1(z^2)$ will remain a FIR biorthogonal partner of $\mathbf{F}(z)$ for any FIR choice of $\mathbf{A}(z)$. This approach gives us infinite freedom to choose $\mathbf{A}(z)$ and optimize the FSE performance.

Example. To demonstrate this we took a 3×3 oversampled FIR channel matrix $\mathbf{F}(z)$ with oversampling factor $M = 2$. The matrix $\mathbf{F}(z)$ can be found in [33]. The normally sampled version $\mathbf{C}(z) = [\mathbf{F}(z)]_{\downarrow 2}$ has an IIR inverse with two poles close to the unit circle. Assume first this IIR equalizer $\mathbf{D}(z)$ is used in the SSE configuration. For a 16-QAM constellation with signal to noise ratio of 18 dB, the results of equalization are as shown in Fig. 12 (top figure; measurements averaged over all three channels). Evidently the IIR filter has amplified the noise to a point where the symbols are indistinguishable. Next consider the FSE scheme of Fig. 11(b) with $M = 2$. In this example there exists an FIR partner $\mathbf{H}(z)$ and the result of equalization is shown in Fig. 12 (middle). This is clearly an improvement. We can improve this further by exploiting the nonuniqueness of the biorthogonal partner $\mathbf{H}(z)$. We did this by constraining $\mathbf{A}(z)$ to be a third order FIR matrix and theoretically optimizing it for fixed noise statistics. This results in the much improved performance shown in Fig. 12 (bottom).

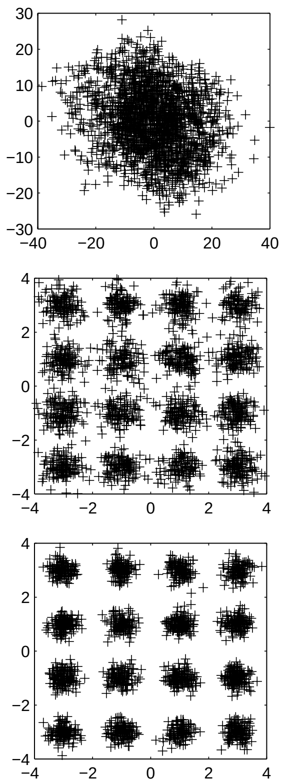

Fig. 12. Channel equalization example with a 16-QAM constellation. IIR equalizer (top), FIR fractionally spaced equalizer $\mathbf{H}(z)$ (middle), and optimized 3rd order FIR fractionally spaced equalizer $\widehat{\mathbf{H}}(z)$ (bottom).

4. NONUNIFORM SAMPLING

In the multiresolution computation of Sec. 3.3, both IIR filtering and oversampling schemes can be *avoided completely* if the samples of $x(t)$ are allowed to be **nonuniform**. Here we will show how to reconstruct $c(n)$ from periodically nonuniform samples of $x(t)$ with the help of FIR filters alone. For example, suppose we consider the following three sets of samples

$$x(3n), \quad x(3n+\frac{1}{3}), \quad x(3n+\frac{2}{3})$$

This is equivalent to the nonuniform sampling scheme shown in Fig. 13, with average rate still equal to unity. The three sets of samples can be expressed as

$$x(3n+\frac{i}{3}) = \sum_k c(k)\phi(3n+\frac{i}{3}-k), \quad i = 0,1,2.$$

Thus we can represent the set of nonuniform samples as the decimated subband signals of a maximally decimated analysis bank (Fig. 14(a)) where $H_i(z)$ have impulse responses

$$h_i(n) = \phi(n+\frac{i}{3}), \quad i = 0,1,2. \tag{13}$$

If there exists an FIR synthesis filter bank with the perfect reconstruction property [22] then we can reconstruct $c(n)$ from these nonuniform samples (Fig. 14(b)), and recover $x(t) = \sum_n c(n)\phi(t-n)$.

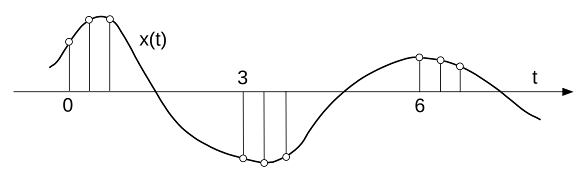

Fig. 13 . Demonstration of nonuniform sampling

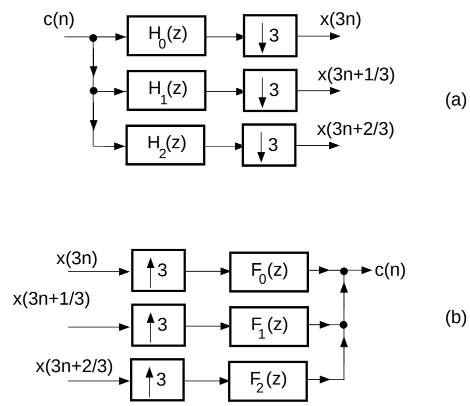

Fig. 14. (a) Analysis bank representing nonuniform sampling, and (b) synthesis bank for reconstructing $c(n)$.

Consider the case where $\phi(t)$ is the quadratic spline

$$\phi(t) = \begin{cases} t^2/2 & 0 \le t < 1 \\ 3/4 - (t-3/2)^2 & 1 \le t < 2 \\ (t-3)^2/2 & 2 \le t < 3 \\ 0 & \text{otherwise.} \end{cases} \tag{14}$$

Then the analysis filters are

$$H_0(z) = \frac{z^{-1}+z^{-2}}{2}, \quad H_1(z) = \frac{1+13z^{-1}+4z^{-2}}{18},$$

and $H_2(z) = z^{-2}H_1(z^{-1})$. Using standard techniques from filter bank theory [22] we can obtain the synthesis filters for perfect reconstruction:

$$F_0(z) = \frac{13-5z+13z^2}{4}, \quad F_1(z) = \frac{-36+12z-12z^2}{4},$$

and $F_2(z) = (27 - 3z + 3z^2)/4$. Thus, $c(n)$ can be reconstructed from the nonuniform samples of $x(t)$ using these FIR filters, as shown in Fig. 14(b). By using $x(t) = \sum_k c(k)\phi(t-k)$ we can directly express $x(t)$ in terms of the nonuniform samples:

$$x(t) = \sum_{i=0}^{2} \sum_{k=-\infty}^{\infty} x(3k + \frac{i}{3}) s_i(t - 3k) \qquad (15)$$

Here $s_i(t)$ are related to the scaling function $\phi(t)$ and the synthesis filters by $s_i(t) = \sum_k f_i(k)\phi(t-k)$. We can interpret this reconstruction as a continuous-time filter bank as shown in Fig. 15: The three sets of samples are passed through three analog filters $s_0(t), s_1(t)$ and $s_2(t)$ and then added up to obtain $x(t)$. Since $f_i(k)$ have finite durations, the filters $s_i(t)$ also have finite durations like $\phi(t)$.

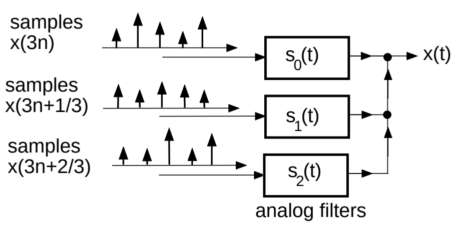

Fig. 15. Reconstruction of $x(t)$ from nonuniform samples.

5. DERIVATIVE SAMPLING

We have seen that signals of the form $x(t) = \sum_k c(k)\phi(t-k)$ can be recovered from integer-rate samples $x(n)$ most of the time. Suppose we have available both $x(t)$ and the first derivative $\dot{x}(t)$. The samples obtained at **half the rate** are given by

$$x(2n) = \sum_k c(k)\phi(2n-k), \quad \dot{x}(2n) = \sum_k c(k)\dot{\phi}(2n-k)$$

The total number of samples per unit time is unity as before. We can regard these as the outputs of the two channel filter bank shown in Fig. 16 where

$$H_0(z) = \sum_n \phi(n) z^{-n}, \qquad H_1(z) = \sum_n \dot{\phi}(n) z^{-n}.$$

Under some conditions on these filters, we can recover $c(n)$ from $x(2n)$ and $\dot{x}(2n)$ perfectly. Once this is done, $x(t)$ can be reconstructed from $x(t) = \sum_k c(k)\phi(t-k)$. An advantage is that we can often obtain FIR reconstruction (i.e., make $F_0(z), F_1(z)$ FIR) as we shall demonstrate. Derivative sampling theorems can be useful, for example, when we have measurements of position and velocity of a moving target or car.

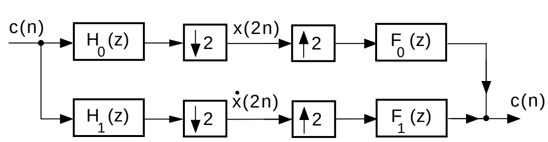

Fig. 16. Two channel filter bank for derivative sampling.

For example consider the case where $\phi(t)$ is the quadratic B-spline given by (14). In this case $\phi(1) = \phi(2) = 0.5$ and $\phi(n) = 0$ otherwise, so that $\Phi_d(z) = 0.5z^{-1}(1 + z^{-1})$. Recovery of $c(n)$ from the full-rate samples $x(n)$ requires the IIR filter $2z/(1+z^{-1})$ which is unstable (pole at $z = -1$). Now consider derivative sampling. We have

$$H_0(z) = 0.5z^{-1}(1 + z^{-1}).$$

The quadratic spline is continuously differentiable once, and the samples of the result are

$$\dot{\phi}(1) = -\dot{\phi}(2) = 1, \quad \text{and} \quad \dot{\phi}(n) = 0 \text{ otherwise},$$

so that $H_1(z) = z^{-1}(1 - z^{-1})$. The synthesis filters which give perfect reconstruction are uniquely given by

$$F_0(z) = z(1+z) \quad \text{and} \quad F_1(z) = z(1-z)/2 \qquad (16)$$

These are simple FIR filters indeed! Using these filters we can express $c(n)$ in terms of $x(2n)$ and $\dot{x}(2n)$. Substituting into $x(t) = \sum c(n)\phi(t-n)$ we finally obtain

$$x(t) = \sum_i x(2i)s_0(t-2i) + \sum_i \dot{x}(2i)s_1(t-2i)$$

where $s_0(t)$ and $s_1(t)$ are given by

$$\begin{aligned} s_0(t) &= \phi(t+1) + \phi(t+2) \\ s_1(t) &= 0.5\phi(t+1) - 0.5\phi(t+2). \end{aligned}$$

The idea of FIR reconstruction from derivative samples can be extended for higher order splines [30], [21].

6. DISCRETE TIME CASE

If the Fourier transform of a discrete time signal $x(n)$ is restricted to $|\omega| < \sigma < \pi$ we say that $x(n)$ is bandlimited (σ-BL). If $x(n)$ is bandlimited to π/M we can decimate it by M without aliasing.[5] Even if this is not the case (e.g., if $X(e^{j\omega})$ is nonzero everywhere) it is sometimes still possible to recover $x(n)$ from the decimated samples $x(nM)$. This is analogous to the case of non bandlimited continuous time signals (1) which can be recovered from samples.

6.1. Discrete-Time Non-BL Sampling

The discrete-time analog of the signal model (1) is

$$x(n) = \sum_k c(k) f(n - kM)$$

where $f(n)$ is a digital filter taking the role of $\phi(t)$. This equation means that $x(n)$ can be represented as the output of an interpolation filter driven by $c(n)$ (Fig. 17).

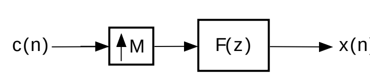

Figure 17. Signal model allowing reconstruction from samples.

[5]All quantities in this section are in discrete-time; the subscript d is therefore avoided for simplicity.

If the filter $f(n)$ has the Nyquist(M) property $f(Mn) = \delta(n)$, then it is clear that $x(Mn) = c(n)$. Thus the undersampled or decimated version $x(Mn)$ can be used to recover $x(n)$ completely. Even when this Nyquist condition is not valid, it is still possible to recover $x(n)$ from the undersampled version $x(Mn)$ under some mild conditions on the model filter $F(z)$ [23, 26]. To see this note that

$$x(Mn) = \sum_k c(k)f(M(n-k)) = \sum_k c(k)r_0(n-k)$$

where $r_0(n) = f(Mn)$. Defining $R_0(z)$ to be the filter with impulse response $r_0(n)$, we see that $c(n)$ can be recovered as the output of $1/R_0(z)$ in response to the input $x(Mn)$. Thus, from $x(Mn)$ we can find $c(n)$, and use $x(n) = \sum_k c(k)f(n-Mk)$ to find $x(n)$.

This idea succeeds as long as $R_0(z)$ has no unit circle zeros. Notice that $R_0(z)$ is nothing but the 0th polyphase component of $F(z)$ in the polyphase representation $F(z) = \sum_{i=0}^{M-1} z^i R_i(z^M)$. As long as there exists one polyphase component $R_i(z)$ free from unit circle zeros, we can use it to write

$$x(Mn-i) = \sum_k c(k)f(M(n-k)-i) = \sum_k c(k)r_i(n-k).$$

Then $c(k)$ can be recovered from the samples $x(Mn-i)$ by filtering through $1/R_i(z)$, and then $x(n)$ can be found from Fig. 17.

6.2. Difference-Sampling

If none of the polyphase components of $F(z)$ is free from unit circle zeros, then we can try other combinations of samples. Thus let $M = 2$ and

$$F(z) = 1 + z - z^2 + z^3 = 1 - z^2 + z(1+z^2)$$

which shows $R_0(z) = 1 - z$ and $R_1(z) = 1 + z$. Both of these have unit circle zeros. So we cannot recover $x(n)$ from the two-fold decimated version $x(2n)$ or from $x(2n-1)$ in a stable manner. Now consider the signal $x(n)$ and its first difference $x(n) - x(n-1)$. If we decimate these by **four** we get

$$x_0(n) = x(4n), \quad x_1(n) = x(4n) - x(4n-1).$$

Together, these two signals still imply an average two-fold decimation. We will show that $x(n)$ can be recovered from $x_0(n)$ and $x_1(n)$ using stable, in fact FIR, filters. This is analogous to the derivative sampling scheme of Sec. 5. To prove the preceding claim note that the signal model is $X(z) = C(z^2)F(z)$ so that

$$\begin{aligned} X_0(z) &= \Big[C(z)[F(z)]_{\downarrow 2}\Big]_{\downarrow 2} \\ X_1(z) &= \Big[C(z)\Big[(1-z^{-1})F(z)\Big]_{\downarrow 2}\Big]_{\downarrow 2} \end{aligned}$$

Defining $H_0(z) = [F(z)]_{\downarrow 2} = 1 - z$ and

$$H_1(z) = [(1-z^{-1})F(z)]_{\downarrow 2} = -2z,$$

we see that the samples $x_0(n)$ and $x_1(n)$ can be represented as the outputs of an analysis bank (Fig. 18). Using the synthesis filters

$$F_0(z) = 1, \quad F_1(z) = -(1+z^{-1})/2$$

we verify that this is a perfect reconstruction system. In short, $c(n)$ can be recovered from the samples $x(4n)$ and $x(4n) - x(4n-1)$ using the FIR filters $F_0(z)$ and $F_1(z)$. The original signal $x(n)$ can then be recovered from the basic model of Fig. 17. This is analogous to the derivative sampling theorem given in Sec. 5.

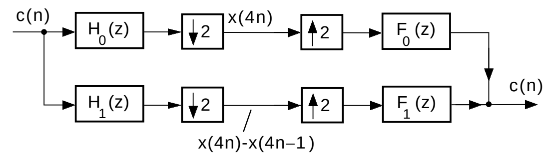

Fig. 18. Two channel filter bank for difference-sampling.

6.3. Least Squares Modelling

The relation between the model of Fig. 17 and the concept of biorthogonal partners arises as follows: Suppose $F(z)$ has a biorthogonal partner $H(z)$ (with respect to M). Then we can recover the driving signal $c(n)$ from $x(n)$ by using the decimation filter shown in Fig. 19. This follows from the fact that if we cascade this figure to the right of Fig. 17, then the result is an identity system by definition of biorthogonal partners (compare with Fig. 6).

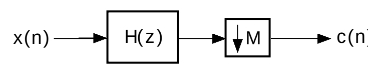

Fig. 19. Recovery of the driving signal $c(n)$.

Assume next that we have an arbitrary discrete time signal $x_{arb}(n) \in \ell_2$ which we wish to model as the output of a fixed interpolation filter $F(z)$. In general there may not exist $c(n)$ such that the output of Fig. 17 is exactly $x_{arb}(n)$. One approach would be to find $c(n)$ such that the error between $x_{arb}(n)$ and the actual output $x(n)$ of $F(z)$ minimized. More specifically let us say that

$$\sum_n |x_{arb}(n) - x(n)|^2$$

should be minimized. It can be shown [25] that the driving signal $c(n)$ which generates this optimal $x(n)$ is precisely the output of the decimation filter Fig. 19 in response to the input $x_{arb}(n)$, where $H(z)$ is given by

$$H(e^{j\omega}) = \frac{F^*(e^{j\omega})}{[|F(e^{j\omega})|^2]_{\downarrow M \uparrow M}}$$

It is readily verified that this $H(z)$ is a biorthogonal partner of $F(z)$. We call it the **least squares** biorthogonal partner for obvious reasons. Please see [25] for further details. Since the output of $F(z)$ in Fig. 17 has the form

$x(n) = \sum_k c(k)f(n - kM)$, it belongs to the space $\mathcal{F}_M$ spanned by the sequences

$$\eta_k(n) \stackrel{\Delta}{=} f(n - kM)$$

Assuming that all signals and filters are in ℓ_2, the above optimization has a nice interpretation: the best approximation $x(n)$ described above is nothing but the projection of $x_{arb}(n)$ onto $\mathcal{F}_M$.

7. CONCLUDING REMARKS

The interpretation of sampling theorems in terms of digital filters and filter banks has led to new insights, unifications, and new attractive implementations. In addition several new applications have been opened up as well. In this lecture we presented an overview of these. One application we have not mentioned is the oversampling of bandlimited signals for noise shaping as in Σ-Δ modulators. A similar idea works for non BL signals as well [16].

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